

# CONTRACTIBILITY OF MANIFOLDS BY MEANS OF STOCHASTIC FLOWS

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**ABSTRACT.** In the paper [Probab. Theory Relat. Fields, **100** (1994) 417–428] Xue-Mei Li studied stability of stochastic differential equations and the interplay between the moment stability of a SDE and the topology of the underlying manifold. In particular, she gave sufficient condition on SDE on a manifold  $M$  under which the fundamental group  $\pi_1 M = 0$ . We prove that in fact under the same conditions the manifold  $M$  is contractible, that is all homotopy groups  $\pi_k M$ ,  $k \geq 1$ , vanish. The proof follows the arguments of Xue-Mei Li.

## 1. INTRODUCTION

Let  $M$  be a smooth connected manifold possibly with boundary,  $\mathcal{T} = (\Omega, \mathcal{F}, \mathbf{P})$  be a probability space. In this paper by a *stochastic flow* we will mean a *complete homogeneous in time stochastic flow of homeomorphisms* which is a map

$$(1) \quad \xi : M \times [0, +\infty) \times \Omega \rightarrow M$$

having the following properties: there exists a subset  $N \subset \Omega$  of measure 0 such that for all  $x \in M$ ,  $s, t \geq 0$ , and  $\omega \in \Omega \setminus N$

- a) the map  $\xi_{t,\omega} : M \rightarrow M$ ,  $\xi_{t,\omega}(x) = \xi(x, t, \omega)$ , is a homeomorphism of  $M$ ,
- b)  $\xi_{0,\omega}(x) = x$ ,
- c)  $\xi_{t,\omega}(\xi_{s,\omega}(x)) = \xi_{s+t,\omega}(x)$ .

It is well known that for a large class of SDE their solutions are stochastic flows, however a priori, not every stochastic flow is a solution of certain SDE. For details on this correspondence, see e.g. [4, Chapter 4].

In the paper [5] Xue-Mei Li studied moment stability of SDE of the form

$$(2) \quad dx_t = X(x_t) \circ dB_t + A(x_t)dt,$$

where  $B_t$  is an  $m$ -dimensional Brownian motion on  $\mathcal{T}$ ,  $A$  is a vector field on  $M$ , and  $X \in \text{Hom}(\mathbb{R}^n, TM)$  is a bundle homomorphism of class  $C^3$  from trivial  $\mathbb{R}^n$ -bundle  $\underline{\mathbb{R}^n} = \mathbb{R}^n \times M \rightarrow M$  over  $M$  to its tangent bundle  $TM \rightarrow M$ . Among other results, Xue-Mei Li gave sufficient conditions for triviality of the fundamental group  $\pi_1 M$  of the manifold  $M$  in terms of coefficients of SDE (2).

The aim of the present note is to show that the proof of Theorem 4.1 in [5] contains a surprisingly stronger result about topological structure of  $M$ : in fact *all*(!) of its homotopy groups  $\pi_n M$  vanish, so  $M$  is even *contractible*.

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In particular, the stochastically-geometric assumptions on  $h$ -Brownian motion in Corollaries 4.2 and 4.3 of Xue-Mei Li's paper [5] also imply contractibility of  $M$ .

Since the main result of our paper is essentially topological, we want to draw attention of the corresponding audience and so preserve more detailed notation than usually used in stochastic theory.

Till the end of this section we will discuss the SDE corresponding to our flows. The main result will be formulated and proven in Section 3.

**Remark 1.** Consider the general form SDE:

$$(3) \quad d\xi_t(\omega) = X(\xi_t(\omega)) \circ z_t(\omega),$$

where  $z : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^n$  is a random process, and  $X : \text{Hom}(\mathbb{R}^n, TM)$  is a bundle homomorphism from trivial  $\mathbb{R}^n$ -bundle over  $M$  to its tangent bundle  $TM$ , so we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n = \mathbb{R}^n \times M & \xrightarrow{X} & TM \\ & \searrow p_2 \quad \swarrow q & \\ & M & \end{array}$$

in which  $p_2$  is the projection to the second coordinate, and  $q$  is a tangent bundle projection, and for each  $x \in M$  the map  $X_x : \mathbb{R}^n \rightarrow T_x M$  is linear, see e.g. [2, Ch. VII, §1].

Usually, the solution of SDE defines a **local flow**, i.e. there exists a subset  $N \subset \Omega$  of measure 0 such that for all  $\omega \in \Omega \setminus N$  and all  $x \in M$  the trajectory  $\xi_{x,\omega}(t)$  is defined only on some interval  $[0, a_x)$ , for some  $a_x \in (0, +\infty]$  called **explosion time**. If  $a_x = \infty$  for all  $\omega \in \Omega \setminus N$  and all  $x \in M$ , then the SDE and the corresponding flow  $\xi$  is said to be **complete**.

For instance, if  $X$  satisfies **global Lipschitz** condition, then the flow is complete, [2, Chapter 7, Corollary 6.1].

**Remark 2.** Suppose that a stochastic flow (1) is a solution of SDE (3). Suppose also that  $M$  has a non-empty boundary. Then assumption a) implies that for almost all  $\omega \in \Omega$  and all  $s \in [0, +\infty)$  the homeomorphism  $\xi_{s,\omega}$  maps the boundary  $\partial M$  to itself. So  $\partial M$  is **invariant with respect to  $\xi$** .

We want express this property in terms of the corresponding SDE.

Notice that for every  $(v, x) \in \mathbb{R}^n \times M$  its image  $X(v, x)$  belongs to the tangent space  $T_x M$  of  $M$  at  $x$ , so for a fixed  $v \in \mathbb{R}^n$  the map  $X_v : M \rightarrow TM$  can be regarded as a vector field on  $M$ .

Then by [2, Chapter VII, §3, Theorem 3] the invariance of  $\partial M$  with respect to  $\xi$  is equivalent to the statement that **for every  $v \in \mathbb{R}^n$  the vector field  $X_v$  is tangent to  $\partial M$** , i.e. for each  $x \in \partial M$  we have that  $X_v(x) \in T_x \partial M$ .

In the case of SDE (2)  $X$  and  $A$  must be tangent to  $\partial M$ .

## 2. CONTINUITY OF STOCHASTIC FLOWS AND HOMOTOPY

Let  $\xi$  be a complete homogeneous in time stochastic flow which is also a solution of an SDE (3). Suppose also that  $\xi$  has the following properties: there exists a subset  $N \subset \Omega$  of measure 0 such that for all  $\omega \in \Omega \setminus N$

- (i) for each  $x \in M$ , the orbit map  $[0, +\infty) \rightarrow M$  defined by  $t \mapsto \xi_{t,\omega}(x)$  is continuous (*sample path continuity*), and
- (ii) the map  $\xi_{t,\omega} : M \rightarrow M$  is a  $C^1$  diffeomorphism of  $M$ .

These properties hold under wide assumptions on  $X$ , e.g. when  $X$  is smooth, see [2, Chapter VII, Corollary 6.2, & Chapter VIII, Corollaries 1C.1-3].

Recall that two continuous maps  $f, g : A \rightarrow B$  between topological spaces  $A$  and  $B$  are called *homotopic*, if there exists a continuous map  $F : A \times [a, b] \rightarrow B$ , that is continuous in  $(x, t) \in A \times [a, b]$ , such that  $F_a = f$  and  $F_b = g$ . If  $f$  and  $g$  are homotopic, we write

$$f \simeq g.$$

The essential point in the proof of main result is the existence of a homotopy between  $\xi_{s,\omega}$  and  $\xi_{t,\omega}$  for all  $s, t \in [0, \infty)$ .

This holds e.g. when  $\xi_\omega : M \times [0, +\infty) \rightarrow M$  is joint continuous in  $(x, t)$ . However, due to (i) and (ii), this map is in general only *separately* continuous in  $x$  and  $t$ .

Nevertheless, the following lemma gives sufficient conditions for existence of homotopy of  $\xi_{t,\omega}$  and  $\xi_{s,\omega}$  restricted to compact subsets of  $M$ . This statement will suffice for our purposes.

Let  $\rho$  be a Riemannian metric on  $M$ . For each  $x \in M$  denote by  $R_x$  the injective radius at  $x$ , that is the largest radius for which the exponential map at  $x$  is a diffeomorphism. For a subset  $K \subset M$  let

$$(4) \quad R_K = \inf_{x \in K} R_x.$$

If  $K$  is compact, then  $R_K > 0$ .

**Lemma 1.** *Let  $\xi$  be a stochastic flow on  $M$ ,  $K \subset M$  a compact subset, and  $W$  a neighbourhood of  $K$ . For  $x \in K$  let the random variable  $\tau_{x,W} : \Omega \rightarrow [0, +\infty)$  be the first exit time from  $W$  of the trajectory  $t \mapsto \xi_\omega(x, \cdot)$  started from  $x$  at  $t = 0$ .*

*Suppose that the following condition holds true: for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$(5) \quad \mathbf{P}(\omega \in \Omega \mid \exists x \in K : \tau_{x,W}(\omega) \leq \delta) \leq \varepsilon.$$

*Then for almost all  $\omega \in \Omega$  and for any  $t \geq 0$  the restriction  $\xi_{t,\omega}|_K : K \rightarrow M$  of  $\xi_{t,\omega}$  to  $K$  is homotopic to the identity inclusion  $\xi_{0,\omega}|_K : K \subset M$ .*

**Remark 3.** *Condition (5) holds, for example, for solutions of SDE with  $C^2$  coefficients, see [2, Chapter VII, §5, Corollary 5].*

*Proof.* Fix any Riemannian metric on  $M$ . We can assume (due to compactness of  $K$ ) that  $W$  is a  $b$ -neighbourhood of  $K$  for some  $b < R_K$ , so for every  $x \in K$ , the ball of radius  $b$  is contained in  $W$ . Let also

$$B = \{\omega \in \Omega \mid \xi_{t,\omega}|_K \simeq \xi_{0,\omega}|_K, \forall t \geq 0\}.$$

We have to show that  $\mathbf{P}(B) = 1$ . It suffices to establish that  $\mathbf{P}(B) > 1 - \varepsilon$  for any  $\varepsilon > 0$ . The proof will be given in three steps.

**Step 1.** First we will show that *there exists  $\delta > 0$  and a subset  $Q \subset \Omega$  such that  $\mathbf{P}(Q) \geq 1 - \varepsilon$  and*

$$(6) \quad \xi_{t,\omega}|_K \simeq \xi_{0,\omega}|_K, \quad \omega \in Q, \quad t \in [0, \delta].$$

By assumption there exists  $\delta > 0$  such that

$$\mathbf{P}(\omega \in \Omega \mid \exists x \in K : \tau_{x,W}(\omega) \leq \delta) < \varepsilon.$$

Put

$$\begin{aligned} Q &= \{\omega \in \Omega \mid \xi_\omega(x, t) \in W, \forall x \in K, 0 \leq t \leq \delta\} \\ &= \{\omega \in \Omega \mid \tau_{x,W}(\omega) > \delta, \forall x \in K\}. \end{aligned}$$

Then

$$\mathbf{P}(Q) = 1 - \mathbf{P}(\omega \in \Omega \mid \exists x \in K : \tau_{x,W}(\omega) \leq \delta) \geq 1 - \varepsilon.$$

Thus with probability  $\geq 1 - \varepsilon$  each  $x \in K$  belongs to  $W$  for all  $t \in [0, \delta]$ .

It remains to prove (6). For every  $x \in K$ ,  $t \in [0, \delta]$  and  $\omega \in Q$  there exists a unique geodesic  $\gamma_{x,t,\omega} : [0, 1] \rightarrow W$  of length  $< R_K$  connecting  $x$  and  $\xi_{t,\omega}(x)$ , that is

$$(7) \quad \gamma_{x,t,\omega}(0) = x, \quad \gamma_{x,t,\omega}(1) = \xi_{t,\omega}(x).$$

Moreover, since a geodesic continuously depends on its ends, for each  $t \in [0, \delta]$  the following map

$$\Gamma_{t,\omega} : K \times [0, 1] \rightarrow W, \quad \Gamma_{t,\omega}(x, s) = \gamma_{x,t,\omega}(s),$$

is joint continuous in  $(x, s)$ . Therefore this map is a homotopy between  $\text{id}_K$  and the restriction of  $\xi_{t,\omega}$  to  $K$ .

**Step 2.** Now we show that *for all  $\omega \in Q$  and  $t, t' \in [0, +\infty)$  such that  $0 \leq t - t' \leq \delta$ , the restrictions of  $\xi_{t,\omega}$  and  $\xi_{t',\omega}$  to  $K$  are homotopic.*

By the property c) of the stochastic flow we get

$$\xi_{t',\omega} = \xi_{t,\omega} \circ \xi_{t-t',\omega}.$$

Since  $t - t' \in [0, \delta]$ , it follows from Step 1 that  $\xi_{t-t',\omega}|_K$  is homotopic to  $\text{id}_K$ . Hence  $\xi_{t',\omega}$  and  $\xi_{t,\omega}$  are homotopic as well.

**Step 3.** Finally we prove that  $\xi_{t,\omega}|_K$  is homotopic to  $\xi_{0,\omega}|_K$  for any  $\omega \in Q$  and  $t \in [0, +\infty)$ .

Notice that  $n\delta \leq t < (n+1)\delta$  for some integer  $n \geq 0$  and by Step 2 in the following sequence of restrictions to  $K$

$$\xi_{0,\omega}|_K, \quad \xi_{\delta,\omega}|_K, \quad \xi_{2\delta,\omega}|_K, \quad \dots, \quad \xi_{n\delta,\omega}|_K, \quad \xi_{t,\omega}|_K$$

any two consecutive maps are homotopic. Therefore all these maps are homotopic each other.

Steps 1-3 imply that  $Q \subset B$  and so  $\mathbf{P}(B) \geq \mathbf{P}(Q) \geq 1 - \varepsilon$ . Since  $\varepsilon$  is arbitrary, we get that  $\mathbf{P}(B) = 1$ . Lemma 1 is completed.  $\square$

### 3. MAIN RESULT

**Theorem 1.** *Let  $M$  be a smooth connected manifold and  $\xi$  be a complete homogeneous in time stochastic flow satisfying conditions (i) and (ii) of Section 2. Suppose that the following conditions hold true.*

- (a) *There exists a subset  $N' \subset \Omega$  of measure 0 such that for all  $\omega \in \Omega \setminus N'$ , any  $t \geq 0$  and any compact subset  $K \subset M$  the restriction of  $\xi_{t,\omega}$  to  $K$  is homotopic to the identity inclusion  $\xi_{0,\omega}|_K : K \subset M$ .*
- (b) *For any compact subset  $\mathbf{L}$  of the tangent bundle  $TM$  we have that*

$$(8) \quad \int_0^{+\infty} \sup_{(x,v) \in \mathbf{L}} \mathbf{E} |T_x \xi_t \cdot v| dt < \infty.$$

- (c) *There exists a measure  $\mu$  on  $M$  such that for any compact subset  $K \subset M$  and  $x \in K$  we have that*

$$\lim_{t \rightarrow \infty} \mathbf{P}(\xi_t(x) \in K) = \mu(K).$$

*Then  $M$  is contractible.*

**Remark 4.** 1) Condition (a) holds e.g. when the flow satisfies assumptions of Lemma 1 which actually give estimate on exit time for stochastic flow  $\xi$ . It replaces the notion of strong 1-completeness used in [5]. Since the flows considered in that paper are solutions of SDE with  $C^2$  coefficients, condition (a) holds for them, and therefore the assumption about strong 1-completeness in [5] is in fact unnecessary.

2) Condition (b) is equivalent to the corresponding condition of [5, Theorem 4.1].

3) Condition (c) is satisfied, in particular, if  $\mu$  is **strongly mixing** with respect to  $\xi$ , see [1, Theorem 3.2.4]. This property is slightly stronger than **ergodicity**.

4) Suppose that in Theorem 1  $M$  is compact. Then  $\partial M \neq \emptyset$ , **and so  $\xi$  is tangent to  $\partial M$** . Indeed, if  $M$  is compact and  $\partial M = \emptyset$  (such manifolds are called **closed**), then it is not contractible, since it has non-trivial cohomology groups:  $H^d(M, \mathbb{Z}) = \mathbb{Z}$ , for orientable  $M$ , and  $H^d(M, \mathbb{Z}_2) = \mathbb{Z}_2$  for non-orientable  $M$ , where  $d = \dim M$ , see [3, Theorem 3.2.6].

*Proof.* The proof follows the line of [5, Theorem 4.1]. We will show that all the homotopy group of  $M$  vanish. Then Whitehead's theorem implies that  $M$  is contractible, e.g. [3, Theorem 4.5].

Let  $S^n$  be a unit sphere in  $\mathbb{R}^{n+1}$ . For any continuous map  $\sigma : S^n \rightarrow M$ , ( $n \geq 1$ ),  $t \in [0, +\infty)$  define the random map  $\sigma_t : \Omega \rightarrow C(S^n, M)$  which for almost all  $\omega \in \Omega$  is defined by

$$\sigma_t(\omega)(x) = \xi_{t,\omega}(\sigma(x)).$$

It will be convenient to denote the map  $\sigma_t(\omega)$  by  $\sigma_{t,\omega}$ .

Since the image  $\sigma(S^n)$  is a compact subset of  $M$ , it follows from (a) that there exists a subset  $N \subset \Omega$  of measure 0 such that for all  $\omega \in \Omega \setminus N$  and any  $t \geq 0$  the restriction of each map  $\xi_{t,\omega}$  to  $\sigma(S^n)$  is homotopic to the identity inclusion  $\xi_{0,\omega} : \sigma(S^n) \subset M$ , and this imply that  $\sigma_{t,\omega} = \xi_{t,\omega} \circ \sigma$  is homotopic to  $\sigma = \xi_{0,\omega} \circ \sigma$ .

Therefore it suffices to show that there exists a subset  $Z \subset \Omega$  of positive measure such that for every  $\omega \in Z$  and some time  $t = t(\omega) > 0$  the image  $\sigma_{t,\omega}(S^n)$  is contained in some geodesic ball. So  $\sigma_{t,\omega}$  is null-homotopic. Then due to  $\mathbf{P}(Z) > 0$  we have that

$$Z \setminus N \neq \emptyset.$$

Therefore for  $\omega \in Z \setminus N$  we will have that  $\sigma_{t,\omega}$  and  $\sigma$  are homotopic. This will imply that  $\sigma$  is null-homotopic as well.

So we have to construct the set  $Z$ .

Since every continuous map  $\sigma : S^n \rightarrow M$  is homotopic to a  $C^1$ -map, we may assume that  $\sigma$  is of class  $C^1$ , whence due to (ii) of Section 2 for almost all  $\omega \in \Omega$  and all  $t \in [0, +\infty)$  the map  $\sigma_{t,\omega} = \xi_{t,\omega} \circ \sigma$  is  $C^1$  as well.

Fix any Riemannian metric on  $M$  and for each  $t \in [0, \infty)$  let

$$\text{diam}(\sigma_t) : \Omega \rightarrow [0, +\infty)$$

be the random variable equal to the diameter of the image of  $\sigma_{t,\omega}(S^n)$  in  $M$ .

**Lemma 2.** *There exists a sequence of numbers  $\{t_j\} \subset [0, +\infty)$  converging to infinity such that*

$$(9) \quad \lim_{j \rightarrow \infty} \mathbf{E}(\text{diam}(\sigma_{t_j})) = 0.$$

*Proof.* We will prove a stronger statement. Recall that by definition

$$\text{diam}(A) = \sup_{x,y \in A} d(x,y).$$

For any pair of distinct points  $x, y \in S^n$  and  $t \in [0, +\infty)$  we can define two random maps

$$\begin{aligned} \bar{x}_t : \Omega &\rightarrow M, & \bar{x}_t(\omega) &= \sigma_{t,\omega}(x), \\ \bar{y}_t : \Omega &\rightarrow M, & \bar{y}_t(\omega) &= \sigma_{t,\omega}(y). \end{aligned}$$

Let also  $e$  be any great circle of radius 1 in  $S^n$  passing through  $x$  and  $y$ . Then we can define the random variable  $l_{t,e} : \Omega \rightarrow [0, +\infty)$  by

$$l_{t,e}(\omega) = \text{length}(\sigma_{t,\omega} \circ e).$$

Evidently,

$$d(\bar{x}_t, \bar{y}_t) \leq \frac{1}{2} l_{t,e},$$

whence

$$\text{diam}(\sigma_t) \leq \frac{1}{2} \sup_{e \text{ is a great circle}} l_{t,e}.$$

Notice that every great circle is uniquely determined by a 2-plane in  $\mathbb{R}^{n+1}$  passing through the origin and vice versa, so the space of great circles in  $S^n$  can be identified with the Grassmannian manifold  $G_2^{n+1}$  of 2-planes in  $\mathbb{R}^{n+1}$ . It is also easy to see that the function  $l_{t,e}(\omega)$  is continuous in  $e$  whenever  $\sigma_{t,\omega}(x)$  is  $C^1$  in  $x$ . Also notice that  $G_2^{n+1}$  is compact. Hence for the proof of (9) it suffices to show that

$$(10) \quad \lim_{j \rightarrow \infty} \sup_{e \in G_2^{n+1}} \mathbf{E} l_{t_j}(e) = 0,$$

for a some sequence  $\{t_j\}$  tending to infinity.

Let  $p : TS^n \rightarrow S^n$  be the tangent bundle of  $S^n$  and  $US^n \subset TS^n$  be the sphere-bundle consisting of all tangent vectors of length 1. Evidently,  $US^n$  is compact. Denote by  $\mathbf{L} = T\sigma(US^n) \subset TM$  the image of  $US^n$  in  $TM$  under the tangent map  $T\sigma : TS^n \rightarrow TM$ .

Regard every great circle  $e$  in  $S^n$  as a map  $e : [0, 2\pi] \rightarrow S^n$  which preserves length. Let also  $\gamma : [0, 2\pi] \rightarrow T[0, 2\pi]$  be unit tangent vector field. Then for almost all  $\omega$  the length  $l_{t,e}(\omega)$  can be calculated via the following formula:

$$\begin{aligned} l_{t,e}(\omega) &= \int_0^{2\pi} |T_s(\xi_{t,\omega} \circ \sigma \circ e)\gamma(s)| ds \\ &= \int_0^{2\pi} |T_{\sigma(e(s))}\xi_{t,\omega} \circ T_s(\sigma \circ e)(\gamma(s))| ds. \end{aligned}$$

Notice that

$$(\sigma \circ e(s), T_s(\sigma \circ e)(\gamma(s))) \in \mathbf{L}.$$

Therefore

$$\begin{aligned} \int_0^{+\infty} \sup_{e \in G_2^{n+1}} \mathbf{E} l_{t,e} dt &= \int_0^{+\infty} \sup_{e \in G_2^{n+1}} \mathbf{E} \left( \int_0^{2\pi} |T_{\sigma(e(s))}\xi_t \circ T_s(\sigma \circ e)(\gamma(s))| ds \right) dt \\ &\leq 2\pi \int_0^{+\infty} \sup_{(x,v) \in \mathbf{L}} \mathbf{E} |T_x \xi_t \cdot v| dt \stackrel{(8)}{<} \infty. \end{aligned}$$

This estimate implies (10) and therefore (9). Lemma 2 is completed.  $\square$   $\square$

Now we can finish the proof of Theorem 1.

Let  $K$  be a compact neighbourhood of the image  $\sigma(S^n)$  having positive  $\mu$ -measure  $\mu(K) > 0$ ,  $R_K$  be the injectivity radius of  $K$ , see (4), and  $\{t_j\}$  be a sequence guaranteed by Lemma 2. Then there exists  $n_1 > 0$  such that for each  $j > n_1$

$$(11) \quad \mathbf{P}(\text{diam}(\sigma_{t_j}) \geq R_K/2) < \mu(K)/4.$$

On the other hand, let  $x \in \sigma(S^n) \subset K$ . Then by (c) we have that

$$\lim_{t \rightarrow \infty} \mathbf{P}(\xi_t(x) \in K) = \mu(K).$$

Hence there exists  $n_2 > 0$  such that for  $t_j > n_2$

$$(12) \quad \mathbf{P}(\xi_{t_j}(x) \in K) > \mu(K)/2.$$

Choose  $j > 0$  so that  $t_j > \max(n_1, n_2)$  and define the following set

$$Z = \{\omega \in \Omega \mid \text{diam}(\sigma_{t_j, \omega}) < R_K/2, \xi_{t_j, \omega}(x) \in K\}.$$

Then for every  $\omega \in Z$  we have that the image of  $\sigma_{t_j, \omega}$  is contained in the geodesic ball of radius  $< R_K$ . Therefore the map  $\sigma_{t_j, \omega} : S^n \rightarrow K \subset M$  is contractible. It remains to show that  $\mathbf{P}(Z) > 0$ . Indeed,

$$\begin{aligned} \mathbf{P}(Z) &= \mathbf{P}(\xi_{t_j}(x) \in K) - \mathbf{P}(\text{diam}(\sigma_{t_j}) \geq R_K/2, \xi_{t_j}(x) \in K) \\ &\geq \mathbf{P}(\xi_{t_j}(x) \in K) - \mathbf{P}(\text{diam}(\sigma_{t_j}) \geq R_K/2) \\ &> \mu(K)/2 - \mu(K)/4 = \mu(K)/4 > 0. \end{aligned}$$

The third line is obtained due to (11) and (12). Theorem 1 is completed.  $\square$

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